A CHARACTERIZATION OF ORTHOGONAL VECTOR FIELDS OVER W*-ALGEBRAS OF TYPE I₂

G.D. LUGOVAYA AND A.N. SHERSTNEV*

ABSTRACT. In the paper we give a characterization of a w^*-continuous orthogonal vector field F over an W^*-algebra $\mathcal{N}$ of type I_2 in terms of reductions F on the center of $\mathcal{N}$. As an application it is obtained a proof of the assertion that an arbitrary w^*-continuous orthogonal vector field over a W^*-algebra of type I_2 is stationary.

Introduction

In [1],[2] Masani studied the integration with respect to orthogonal vector measures defined on rings of the sets. These works obtained an extension in numerous publications in context of the noncommutative measure theory. Some settings of problems and approaches to their solutions may be found in [3], [4,§31]. One of intrinsic problem in this subject is the problem of characterization of Hilbert-valued linear mappings of W^*-algebra preserving the orthogonality property (so-called orthogonal vector fields). In some sense there is a standard (well known) result for the commutative W^*-algebra (see for instance [4, Theorem 31.19]).

In the paper we give a characterization of a w^*-continuous orthogonal vector field F over an W^*-algebra $\mathcal{N}$ of type I_2 in terms of reductions F on the center of $\mathcal{N}$. As an application it is obtained a proof of the assertion that an arbitrary w^*-continuous orthogonal vector field over a W^*-algebra of type I_2 is stationary.

1. Preliminaries

Let $\mathcal{A}$ be a W^*-algebra, and $\mathcal{A}^{\mathrm{pr}}$, $\mathcal{A}^{\mathrm{sa}}$, $\mathcal{A}^{\mathrm{un}}$ denote the sets of orthogonal projections, selfadjoint, unitaries elements in $\mathcal{A}$, respectively. We denote by $\mathrm{rp}\,(x)$ the range projection of $x \in \mathcal{A}^+$. It is the least projection of all projections $p \in \mathcal{A}^{\mathrm{pr}}$ such that $px = x$. In the paper we shall mainly deal with W^*-algebras of type I_2. It is known that any W^*-algebra of type I_2 can be represented in the form $\mathcal{N} = \mathcal{M} \otimes M_2$ where $\mathcal{M}$ is a commutative W^*-algebra and M_2 is the

algebra of all 2×2 matrices over $\mathbb{C}$. So the elements of $\mathcal{N}$ are matrices in the form of $(x_{ij}) = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$ where $x_{ij} \in \mathcal{M}$. We denote by $\mathbf{I}$ and $\mathbf{1}$ the units of algebras $\mathcal{N}$ and $\mathcal{M}$ respectively; $\{\varepsilon_{ij}\}_{i,j=1,2}$ denote the matrix unit of algebra $\mathcal{N}$ in the form

$$\varepsilon_{11} = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & 0 \end{pmatrix}, \ \varepsilon_{12} = \begin{pmatrix} 0 & \mathbf{1} \\ 0 & 0 \end{pmatrix}, \ \varepsilon_{21} = \begin{pmatrix} 0 & 0 \\ \mathbf{1} & 0 \end{pmatrix}, \ \varepsilon_{22} = \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{1} \end{pmatrix}.$$

It is known [5, Prop. 1.18.1] that a commutative W^*-algebra $\mathcal{M}$ may be realized as algebra $L^\infty(\Omega, \nu)$ of all essentially bounded locally ν-measurable functions on a localizable measure space (Ω, ν) (i. e. Ω is direct sum of finite measure spaces, see [6]). In this case, the Banach space $L^1(\Omega, \nu)$ is the predual of $L^\infty(\Omega, \nu)$. Now we shall identify $\mathcal{M}$ with $L^\infty(\Omega, \nu)$. In this case the characteristic functions

$$\pi(\omega) \equiv \chi_\pi(\omega) = \begin{cases} 1, & \text{if } \omega \in \pi, \\ 0, & \text{if } \omega \notin \pi, \end{cases} \qquad \pi \subset \Omega.$$

correspond to projections $\pi \in \mathcal{M}^{\mathrm{pr}}$. (We use the same letter π to designate three objects: a projection in $\mathcal{M}^{\mathrm{pr}}$, a ν-measurable set in Ω and the characteristic function of this set.)

Accordingly, the W^*-algebra $\mathcal{N} = \mathcal{M} \otimes M_2$ is realized as the algebra of 2×2-matrices (x_{ij}), $x_{ij} \in L^\infty(\Omega, \nu)$ with the predual

$$\mathcal{N}_* = L^1(\Omega, \nu) \otimes (M_2)_* = L^1(\Omega, \nu, M_2) \tag{1}$$

where $L^1(\Omega, \nu, M_2)$ is the Banach space of all M_2-valued Bochner ν-integrable functions on Ω [5, Prop. 1.22.12]. Elements of the space M_2 on the right-hand side of (1) can be viewed as density matrices. In other words, elements φ of $\mathcal{N}_*$ are the matrices

$$\varphi = \begin{pmatrix} \varphi_{11} & \varphi_{12} \\ \varphi_{21} & \varphi_{22} \end{pmatrix}, \quad \varphi_{ij} \in L^1(\Omega, \nu), \tag{2}$$

and the action of φ on an element $a = (a_{ij})$, $a_{ij} \in L^\infty(\Omega, \nu)$, is given by the equality

$$\varphi(a) = \int_\Omega \mathrm{Tr}\left[(\varphi_{ij})(a_{ij})\right] d\nu = \int_\Omega (\varphi_{11}a_{11} + \varphi_{12}a_{21} + \varphi_{21}a_{12} + \varphi_{22}a_{22}) \, d\nu.$$

In this case

$$\varphi(\pi \varepsilon_{ij}) = \int_\pi \varphi_{ji} \, d\nu, \quad \pi \in \mathcal{M}^{\mathrm{pr}}, \ i,j = 1,2, \, [1]$$

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[1] We use the notations of type $a\varepsilon_{12} \equiv \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}$, $a \in \mathcal{M}$.

i. e. $\varphi_{ij} = \frac{d}{d\nu}\varphi((\cdot)\varepsilon_{ji})$ is the Radon-Nikodym derivative of the charge $\pi \to \varphi(\pi\varepsilon_{ji})$, $\pi \in \mathcal{M}^{\mathrm{pr}}$, with respect to measure ν.

Let H be a complex Hilbert space. A bounded linear map $F : \mathcal{A} \to H$ is said to be *an orthogonal vector field (OVF)* if

$$pq = 0 \ (p, q \in \mathcal{A}^{\mathrm{pr}}) \Rightarrow \langle F(p), F(q) \rangle = 0. \tag{3}$$

If a linear map F with the property (3) is continuous in w^*-topology on $\mathcal{A}$ and weak topology on H, then the map F is referred to as w^*-*continuous OVF*. It should be noted that a w^*-continuous OVF F is bounded, so that it is an OVF [7, Corollary 3]. A w^*-continuous OVF F is said to be *stationary* [3], if there exist two functionals $\varphi, \psi \in \mathcal{A}_*^+$ such that

$$\langle F(x), F(y) \rangle = \varphi(y^*x) + \psi(xy^*), \quad x, y \in \mathcal{A}. \tag{4}$$

Given OVF $F : \mathcal{A} \to H$ assign a positive linear functional $\varrho \in \mathcal{A}^*$,

$$\varrho(x) \equiv \langle F(x), F(1) \rangle, \quad x \in \mathcal{A} \tag{5}$$

($\varrho \in \mathcal{A}_*^+$ as soon as F is w^*-continuous). The following equalities hold

$$\|F(x)\|^2 = \varrho(x^2), \quad x \in \mathcal{A}^{\mathrm{sa}}, \tag{6}$$

$$\mathrm{Re}\,\langle F(x), F(y) \rangle = \frac{1}{2}\varrho(xy + yx), \quad x, y \in \mathcal{A}^{\mathrm{sa}}. \tag{7}$$

Equality (6) follows from the spectral theorem with regard to (3), and the following computation gives (7):

$$\mathrm{Re}\,\langle F(x), F(y) \rangle = \frac{1}{4}[\langle F(x+y), F(x+y) - F(x-y), F(x-y) \rangle]$$
$$= \frac{1}{4}[\varrho((x+y)^2) - \varrho((x-y)^2)] = \frac{1}{2}\varrho(xy + yx), \quad x, y \in \mathcal{A}^{\mathrm{sa}}.$$

It is easily seen that for a stationary OVF the next equality is valid

$$\varrho = \varphi + \psi. \tag{8}$$

2. Orthogonal vector fields over W^*-algebras of type I_2

Let $\mathcal{N} = \mathcal{M} \otimes M_2$ be a W^*-algebra of type I_2, H be a Hilbert space over $\mathbb{C}$ and $F : \mathcal{N} \to H$ be a H-valued OVF. By the proof of Theorem 31.6 [4], the mappings $F_{ij} : \mathcal{M} \to H$ given by the equalities

$$F_{ij}(a) \equiv F(a\varepsilon_{ij}), \quad a \in \mathcal{M}, \ i, j = 1, 2, \tag{9}$$

are OVFs over $\mathcal{M}$. Moreover,

$$F(x) = \sum_{i,j=1}^{2} F_{ij}(x_{ij}), \quad x = (x_{ij}) \in \mathcal{N}.$$

Since the center of $\mathcal{N}$ is isomorphic to algebra $\mathcal{M}$, we can consider the orthogonal vector fields F_{ij} as reductions F on the center of $\mathcal{N}$.

Proposition 1. *Let $F : \mathcal{N} \to H$ be a OVF and F_{ij} are defined in (9). Then*

   (i) $\langle F_{11}(a), F_{22}(b) \rangle = \langle F_{12}(a), F_{21}(b) \rangle = 0, \quad a, b \in \mathcal{M}$,
   (ii) $\|F_{12}(a)\|^2 + \|F_{21}(a)\|^2 = \|F_{11}(a)\|^2 + \|F_{22}(a)\|^2, \quad a \in \mathcal{M}$,
   (iii) $\langle F_{ij}(a), F_{kl}(b) \rangle = \langle F_{ij}(b^*a), F_{kl}(\boldsymbol{1}) \rangle, \quad a, b \in \mathcal{M}, \ i, j, k, l = 1, 2$,
   (iv) $\langle F_{12}(\pi), F_{11}(\boldsymbol{1}) \rangle = \langle F_{22}(\boldsymbol{1}), F_{21}(\pi) \rangle, \ \langle F_{21}(\pi), F_{11}(\boldsymbol{1}) \rangle = \langle F_{22}(\boldsymbol{1}), F_{12}(\pi) \rangle$,
        $\pi \in \mathcal{M}^{\mathrm{pr}}$.

Proof. Let us verify the second equality in (i). In view of the spectral theorem it suffices to prove the assertion in case $a = \sigma, \ b = \tau$ are projections in $\mathcal{M}^{\mathrm{pr}}$. Let us first assume that $\sigma\tau = 0$. We have

$$\begin{pmatrix} 0 & \sigma \\ 0 & 0 \end{pmatrix} = \frac{1}{4} \sum_{\omega = \pm 1, \pm \mathrm{i}} \omega \begin{pmatrix} \sigma & \overline{\omega}\sigma \\ \omega\sigma & \sigma \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ \tau & 0 \end{pmatrix} = \frac{1}{4} \sum_{\omega} \omega \begin{pmatrix} \tau & \omega\tau \\ \overline{\omega}\tau & \tau \end{pmatrix},$$

(10)

where in right-hand sides of (10) there are linear combinations of projections from $\mathcal{N}$. It now follows in view of (3) that $\sigma\tau = 0 \Rightarrow \langle F_{12}(\sigma), F_{21}(\tau) \rangle = 0$.

Now we suppose that $\sigma = \tau$. By the equalities

$$\begin{pmatrix} 0 & \sigma \\ 0 & 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 0 & \sigma \\ \sigma & 0 \end{pmatrix} + \frac{\mathrm{i}}{2} \begin{pmatrix} 0 & -\mathrm{i}\,\sigma \\ \mathrm{i}\,\sigma & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 \\ \sigma & 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 0 & \sigma \\ \sigma & 0 \end{pmatrix} + \frac{\mathrm{i}}{2} \begin{pmatrix} 0 & \mathrm{i}\,\sigma \\ -\mathrm{i}\,\sigma & 0 \end{pmatrix}$$

we have

$$4\langle F_{12}(\sigma), F_{21}(\sigma) \rangle = 4 \left\langle F \begin{pmatrix} 0 & \sigma \\ 0 & 0 \end{pmatrix}, F \begin{pmatrix} 0 & 0 \\ \sigma & 0 \end{pmatrix} \right\rangle$$

$$= \left\| F \begin{pmatrix} 0 & \sigma \\ \sigma & 0 \end{pmatrix} \right\|^2 + \left\langle F \begin{pmatrix} 0 & -\mathrm{i}\,\sigma \\ \mathrm{i}\,\sigma & 0 \end{pmatrix}, F \begin{pmatrix} 0 & \mathrm{i}\,\sigma \\ -\mathrm{i}\,\sigma & 0 \end{pmatrix} \right\rangle$$

$$- \mathrm{i} \left\langle F \begin{pmatrix} 0 & \sigma \\ \sigma & 0 \end{pmatrix}, F \begin{pmatrix} 0 & \mathrm{i}\,\sigma \\ -\mathrm{i}\,\sigma & 0 \end{pmatrix} \right\rangle + \mathrm{i} \left\langle F \begin{pmatrix} 0 & -\mathrm{i}\,\sigma \\ \mathrm{i}\,\sigma & 0 \end{pmatrix}, F \begin{pmatrix} 0 & \sigma \\ \sigma & 0 \end{pmatrix} \right\rangle$$

$$= \left\| F \begin{pmatrix} 0 & \sigma \\ \sigma & 0 \end{pmatrix} \right\|^2 - \left\| F \begin{pmatrix} 0 & -\mathrm{i}\,\sigma \\ \mathrm{i}\,\sigma & 0 \end{pmatrix} \right\|^2$$

$$+ \mathrm{i} \left\{ \left\langle F \begin{pmatrix} 0 & \sigma \\ \sigma & 0 \end{pmatrix}, F \begin{pmatrix} 0 & -\mathrm{i}\,\sigma \\ \mathrm{i}\,\sigma & 0 \end{pmatrix} \right\rangle + \left\langle F \begin{pmatrix} 0 & -\mathrm{i}\,\sigma \\ \mathrm{i}\,\sigma & 0 \end{pmatrix}, F \begin{pmatrix} 0 & \sigma \\ \sigma & 0 \end{pmatrix} \right\rangle \right\}.$$

In addition (see (6),(7)),

$$\left\| F \begin{pmatrix} 0 & -\mathrm{i}\,\sigma \\ \mathrm{i}\,\sigma & 0 \end{pmatrix} \right\|^2 = \left\langle F \left(\begin{pmatrix} 0 & -\mathrm{i}\,\sigma \\ \mathrm{i}\,\sigma & 0 \end{pmatrix}^2 \right), F(\mathbf{I}) \right\rangle$$

$$= \left\langle F \begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix}, F(\mathbf{I}) \right\rangle = \left\| F \begin{pmatrix} 0 & \sigma \\ \sigma & 0 \end{pmatrix} \right\|^2,$$

The expression in braces vanishes:

$$\{...\} = 2\,\mathrm{Re} \left\langle F \begin{pmatrix} 0 & \sigma \\ \sigma & 0 \end{pmatrix}, F \begin{pmatrix} 0 & -\mathrm{i}\,\sigma \\ \mathrm{i}\,\sigma & 0 \end{pmatrix} \right\rangle$$

$$= \varrho \left[\begin{pmatrix} 0 & \sigma \\ \sigma & 0 \end{pmatrix} \begin{pmatrix} 0 & -\mathrm{i}\,\sigma \\ \mathrm{i}\,\sigma & 0 \end{pmatrix} + \begin{pmatrix} 0 & -\mathrm{i}\,\sigma \\ \mathrm{i}\,\sigma & 0 \end{pmatrix} \begin{pmatrix} 0 & \sigma \\ \sigma & 0 \end{pmatrix} \right] = 0,$$

and the second equality in (i) is established. The equality $\langle F_{11}(a), F_{22}(b) \rangle = 0$ $(a, b \in \mathcal{M})$ is obvious in view of the spectral theorem.

Equality (ii) follows from computation (with regard to (i))

$$\|F_{12}(a)\|^2 + \|F_{21}(a)\|^2 = \|F_{12}(a)\|^2 + \|F_{21}(a^*)\|^2 = \|F_{12}(a) + F_{21}(a^*)\|^2$$

$$= \left\langle F \begin{pmatrix} 0 & a \\ a^* & 0 \end{pmatrix}, F \begin{pmatrix} 0 & a \\ a^* & 0 \end{pmatrix} \right\rangle = \left\langle F \left(\begin{pmatrix} 0 & a \\ a^* & 0 \end{pmatrix}^2 \right), F(\mathbf{I}) \right\rangle$$

$$= \langle F_{11}(a^*a), F_{11}(\mathbf{1}) \rangle + \langle F_{22}(a^*a), F_{22}(\mathbf{1}) \rangle = \|F_{11}(a)\|^2 + \|F_{22}(a)\|^2.$$

We used here (for $G = F_{21}$) the following property of the OVF over commutative W^*-algebra $\mathcal{M}$ (it follows easily from the spectral theorem):

$$\|G(a)\|^2 = \|G(a^*)\|^2, \quad a \in \mathcal{M}.$$

Let us establish (iii). We first note that representations (10) give

$$\sigma\tau = 0 \ (\sigma, \tau \in \mathcal{M}^{\mathrm{pr}}) \Rightarrow \langle F_{ii}(\sigma), F_{12}(\tau) \rangle = \langle F_{ii}(\sigma), F_{21}(\tau) \rangle = 0, \quad i = 1, 2. \ (11)$$

In view of the spectral theorem and boundedness of linear maps F_{ij} it is sufficient to establish (iii) for elements a, b in the form

$$a = \sum_t \mu_t \pi_t, \ b = \sum_t \lambda_t \pi_t \quad (\lambda_t, \mu_t \in \mathbb{C}, \ \pi_t \in \mathcal{M}^{\mathrm{pr}}, \ \pi_t \pi_s = 0 \ (t \neq s)).$$

(Here the sums are finite and $\sum_t \pi_t = \mathbf{1}$.) Taking into account (i) and (11), we have

$$\langle F_{ij}(a), F_{kl}(b) \rangle = \sum_{t,s} \mu_t \overline{\lambda_s} \langle F_{ij}(\pi_t), F_{kl}(\pi_s) \rangle = \sum_t \mu_t \overline{\lambda_t} \langle F_{ij}(\pi_t), F_{kl}(\mathbf{1}) \rangle$$

$$= \langle F_{ij}(\sum_t \mu_t \overline{\lambda_t} \pi_t), F_{kl}(\mathbf{1}) \rangle = \langle F_{ij}(b^*a), F_{kl}(\mathbf{1}) \rangle.$$

(iv). We show first that

$$\langle F_{12}(a\pi)+F_{21}(a^*\pi), F_{11}(\pi)\rangle = \langle F_{22}(\pi), F_{12}(a\pi)+F_{21}(a^*\pi)\rangle, \quad a \in \mathcal{M}, \ \pi \in \mathcal{M}^{\mathrm{pr}}. \tag{12}$$

Define $f = F_{12}(a\pi) + F_{21}(a^*\pi) = F\begin{pmatrix} 0 & a\pi \\ a^*\pi & 0 \end{pmatrix}$. By virtue of (iii) and (7)

$$\langle f, F_{11}(\pi)\rangle = \langle f, F(\mathbf{I})\rangle - [\langle f, F_{22}(\pi)\rangle + \langle F_{22}(\pi), f\rangle] + \langle F_{22}(\pi), f\rangle$$

$$= \langle f, F(\mathbf{I})\rangle - 2\,\mathrm{Re}\,\left\langle F\begin{pmatrix} 0 & a\pi \\ a^*\pi & 0 \end{pmatrix}, F\begin{pmatrix} 0 & 0 \\ 0 & \pi \end{pmatrix}\right\rangle + \langle F_{22}(\pi), f\rangle$$

$$= \varrho(a\pi\varepsilon_{12} + a^*\pi\varepsilon_{21}) - \varrho((a\pi\varepsilon_{12} + a^*\pi\varepsilon_{21}) \cdot \pi\varepsilon_{22} + \pi\varepsilon_{22} \cdot (a\pi\varepsilon_{12} + a^*\pi\varepsilon_{21}))$$

$$+ \langle F_{22}(\pi), f\rangle = \langle F_{22}(\pi), f\rangle,$$

and (12) is established. By setting $a = \mathbf{1}$ and $a = \mathrm{i}\,\mathbf{1}$ in (12), we have

$$\langle F_{12}(\pi) + F_{21}(\pi), F_{11}(\pi)\rangle = \langle F_{22}(\pi), F_{12}(\pi) + F_{21}(\pi)\rangle, \tag{13}$$

$$\mathrm{i}\,\langle F_{12}(\pi) - F_{21}(\pi), F_{11}(\pi)\rangle = -\mathrm{i}\,\langle F_{22}(\pi), F_{12}(\pi) - F_{21}(\pi)\rangle. \tag{14}$$

Multiplying both sides of (14) by i and taking into account (13), we have (iv). This proves the proposition. $\square$

Proposition 2. *Let F_{ij} $(i = 1, 2)$ be OVFs satisfying conditions (iii) and (iv) of Prop. 1. Then equality (12) holds.*

Proof. Because the linear mappings F_{ij} are bounded, it is sufficient to prove (12) for finite sums in the form $a = \sum_s \lambda_s \pi_s$ where $\lambda_s \in \mathbb{C}, \pi_s \in \mathcal{M}^{\mathrm{pr}}, \ \pi_s \pi_t = 0 \ (s \neq t), \sum_s \pi_s = \mathbf{1}$. In view of (iii) and (iv) we have

$$\langle F_{12}(a\pi) + F_{21}(a^*\pi), F_{11}(\pi)\rangle = \sum_s [\lambda_s \langle F_{12}(\pi\pi_s), F_{11}(\mathbf{1})\rangle + \overline{\lambda_s}\langle F_{21}(\pi\pi_s), F_{11}(\mathbf{1})\rangle]$$

$$= \sum_s [\lambda_s \langle F_{22}(\mathbf{1}), F_{21}(\pi\pi_s)\rangle + \overline{\lambda_s}\langle F_{22}(\mathbf{1}), F_{12}(\pi\pi_s)\rangle]$$

$$= \langle F_{22}(\pi), F_{21}(\sum_s \overline{\lambda_s}\pi\pi_s)\rangle + \langle F_{22}(\pi), F_{12}(\sum_s \lambda_s\pi\pi_s)\rangle$$

$$= \langle F_{22}(\pi), F_{21}(a^*\pi) + F_{12}(a\pi)\rangle.$$

The proposition follows. $\square$

Theorem 3. *Let $\mathcal{N} = \mathcal{M} \otimes M_2$ be a W^*-algebra of type I_2, $F_{ij} : \mathcal{M} \to H$ be H-valued OVFs (w^*-continuous OVFs) with the properties $(i) - (iv)$ of Prop. 1. Then the linear mapping $F : \mathcal{N} \to H$ defined by the equalities*

$$F(a\varepsilon_{ij}) \equiv F_{ij}(a), \quad a \in \mathcal{M}, \ i, j = 1, 2,$$

is the OVF (respectively, w^*-continuous OVF).

Proof. Since F is w^*–continuous if and only if F_{ij} are w^*-continuous, it is sufficient to consider the case when F_{ij} be OVFs. We have only to verify the equality

$$pq = 0 \implies \langle F(p), F(q) \rangle = 0, \quad p, q \in \mathcal{N}^{\mathrm{pr}}.$$

We use the canonical representation projections in $\mathcal{N}$. Specifically, every projection $r \in \mathcal{N}^{\mathrm{pr}}$ can be expressed in the form [8, Lemma 1]

$$r = \pi_1 \oplus \pi_2 + p(a, v, \pi), \tag{15}$$

where

$$\pi_1 \oplus \pi_2 \equiv \begin{pmatrix} \pi_1 & 0 \\ 0 & \pi_2 \end{pmatrix}, \quad \pi_i \in \mathcal{M}^{\mathrm{pr}},$$

$$p(a, v, \pi) \equiv \begin{pmatrix} a\pi & v\pi(a(\mathbf{1}-a))^{1/2} \\ v^*\pi(a(\mathbf{1}-a))^{1/2} & (\mathbf{1}-a)\pi \end{pmatrix}, \quad \pi \in \mathcal{M}^{\mathrm{pr}}, v \in \mathcal{M}^{\mathrm{un}}.$$

In addition, $\pi_i \leq \mathbf{1} - \pi \; (i = 1, 2), \; a \in \mathcal{M}, \; 0 \leq a \leq \mathbf{1}, \mathrm{rp} \, a(\mathbf{1}-a)) = \mathbf{1}$. Now we present $p, q \in \mathcal{N}^{\mathrm{pr}}$ in the form of (15):

$$p = \tau_1 \oplus \tau_2 + p(a, v, \tau_3), \quad q = \sigma_1 \oplus \sigma_2 + p(b, w, \sigma_3),$$

where $\tau_i, \sigma_i \in \mathcal{M}^{\mathrm{pr}}, \; 0 \leq a, b \leq \mathbf{1}, \; v, w \in \mathcal{M}^{\mathrm{un}}$. From $pq = 0$ it follows [8, Lemma 2] that

$$\tau_i \sigma_i = \tau_i \tau_3 = \sigma_i \sigma_3 = \tau_3 \sigma_i = \tau_i \sigma_3 = 0, \; w\pi = -v\pi, \; b\pi = (\mathbf{1} - a)\pi, \tag{16}$$

where $i = 1, 2, \; \pi \equiv \sigma_3 \tau_3$. Denoting for brevity $\varkappa(x) \equiv (x(\mathbf{1} - x))^{1/2}$ we have

$$F(p) = F_{11}(\tau_1 + a\tau_3) + F_{12}(v\varkappa(a)\tau_3) + F_{21}(v^*\varkappa(a)\tau_3) + F_{22}(\tau_2 + (\mathbf{1} - a)\tau_3),$$
$$F(q) = F_{11}(\sigma_1 + b\sigma_3) + F_{12}(w\varkappa(b)\sigma_3) + F_{21}(w^*\varkappa(b)\sigma_3) + F_{22}(\sigma_2 + (\mathbf{1} - b)\sigma_3).$$

Taking the scalar product of vectors $F(p)$ and $F(q)$ with regard to (i), (iii), (iv) in Prop. 1 and equalities (16) we have

$$\begin{aligned}
\langle F(p), F(q) \rangle = {}& \langle F_{11}(a\pi), F_{11}((\mathbf{1} - a)\pi) \rangle - \langle F_{11}(a\pi), F_{12}(v\varkappa(a)\pi) \rangle \\
& - \langle F_{11}(a\pi), F_{21}(v^*\varkappa(a)\pi) \rangle + \langle F_{12}(v\varkappa(a)\pi), F_{11}((\mathbf{1} - a)\pi) \rangle \\
& - \|F_{12}(v\varkappa(a)\pi)\|^2 + \langle F_{12}(v\varkappa(a)\pi), F_{22}(a\pi) \rangle \\
& + \langle F_{21}(v^*\varkappa(a)\pi), F_{11}((\mathbf{1} - a)\pi) \rangle - \|F_{21}(v^*\varkappa(a)\pi)\|^2 \\
& + \langle F_{21}(v^*\varkappa(a)\pi), F_{22}(a\pi), \rangle - \langle F_{22}((\mathbf{1} - a)\pi), F_{12}(v\varkappa(a)\pi) \rangle \\
& - \langle F_{22}((\mathbf{1} - a)\pi), F_{21}(v^*\varkappa(a)\pi) \rangle + \langle F_{22}((\mathbf{1} - a)\pi), F_{22}(a\pi) \rangle.
\end{aligned}$$

By Prop. 1(ii), the first and the last summands on the right hand-side of obtained equality are mutually annihilated with the fifth and the eights ones. Grouping

the remained summands and setting $c = va\varkappa(a), \; d = v(\mathbf{1} - a)\varkappa(a)$ we have by (12)

$$
\begin{aligned}
\langle F(p), F(q) \rangle &= [\langle F_{12}(v\varkappa(a)\pi) + F_{21}(v^*\varkappa(a)\pi), F_{22}(a\pi) \rangle \\
&\quad - \langle F_{11}(a\pi), F_{12}(v\varkappa(a)\pi) + F_{21}(v^*\varkappa(a)\pi) \rangle] \\
&\quad + [\langle F_{12}(v\varkappa(a)\pi) + F_{21}(v^*\varkappa(a)\pi), F_{11}((\mathbf{1} - a)\pi) \rangle \\
&\quad - \langle F_{22}((\mathbf{1} - a)\pi), F_{12}(v\varkappa(a)\pi) + F_{21}(v^*\varkappa(a)\pi) \rangle] \\
&= [\langle F_{12}(c\pi) + F_{21}(c^*\pi), F_{22}(\pi) \rangle - \langle F_{11}(\pi), F_{12}(c\pi) + F_{21}(c^*\pi) \rangle] \\
&\quad + [\langle F_{12}(d\pi) + F_{21}(d^*\pi), F_{11}(\pi) \rangle - \langle F_{22}(\pi), F_{12}(d\pi) + F_{21}(d^*\pi) \rangle] = 0.
\end{aligned}
$$

The proof is complete. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

3. The stationarity of OVFs over W*-algebras of type I$_2$

As an application of Theorem 3 we will show that every w^*-continuous OVF over an W^*-algebra of type I_2 is stationary. Let $\mathcal{N} = \mathcal{M} \otimes M_2$ be a W^*-algebra of type I_2, $F : \mathcal{N} \to H$ be a w^*-continuous OVF and the functional ϱ be given by (5). Let us observe some properties of reductions (10). With the above notations of Section 1 we have

$$
\varrho(\pi\varepsilon_{ii}) = \langle F(\pi\varepsilon_{ii}), F(\mathbf{I}) \rangle = \langle F(\pi\varepsilon_{ii}), F(\varepsilon_{ii}) \rangle = \langle F_{ii}(\pi), F_{ii}(\mathbf{1}) \rangle, \qquad (17)
$$

where $i = 1, 2, \; \pi \in \mathcal{M}^{\mathrm{pr}}$. Therefore,

$$
\rho_{ii} = \frac{d}{d\nu} \langle F_{ii}(\cdot), F_{ii}(\mathbf{1}) \rangle, \quad i = 1, 2. \qquad (18)
$$

Similarly,

$$
\rho_{ij} = \frac{d}{d\nu} \langle F_{ji}(\cdot), F_{11}(\mathbf{1}) + F_{22}(\mathbf{1}) \rangle, \quad i, j = 1, 2 \; (i \neq j). \qquad (19)
$$

In addition there are defined functions $r_{ij} \in L^1(\Omega, \nu)^+$ such that

$$
\langle F_{ij}(\pi), F_{ij}(\mathbf{1}) \rangle = \int_\pi r_{ij} \, d\nu, \quad r_{ij} = \frac{d}{d\nu} \langle F_{ij}(\cdot), F_{ij}(\mathbf{1}) \rangle, \quad i, j = 1, 2. \qquad (20)
$$

In this connection

$$
r_{ii} = \varrho_{ii} \; (i = 1, 2), \quad r_{12} + r_{21} = \varrho_{11} + \varrho_{22} \; \text{ a. e.} \qquad (21)
$$

(The second equality in (21) follows from Prop. 1(ii).)

If F is stationary there exist (see (4)) $\varphi, \psi \in \mathcal{N}_*^+$ such that

$$
\langle F(x), F(y) \rangle = \varphi(y^*x) + \psi(xy^*), \quad x, y \in \mathcal{N}. \qquad (22)
$$

In this case (8) is satisfied. Let (φ_{ij}) be the matrix of φ (see (2)). We have

$$\int_\pi \varphi_{21} \, d\nu = \varphi(\pi\varepsilon_{12}) = \varphi(\varepsilon_{11}(\pi\varepsilon_{12})\varepsilon_{22}) = \langle F((\pi\varepsilon_{12})\varepsilon_{22}), F(\varepsilon_{11}) \rangle$$

$$= \langle F(\pi\varepsilon_{12}), F(\varepsilon_{11}) \rangle = \langle F_{12}(\pi), F_{11}(\mathbf{1}) \rangle, \quad \pi \in \mathcal{M}^{\mathrm{pr}}.$$

Thus,

$$\varphi_{21} = \frac{d}{d\nu}\langle F_{12}(\cdot), F_{11}(\mathbf{1}) \rangle, \quad \varphi_{12} = \overline{\varphi_{21}} = \frac{d}{d\nu}\langle F_{21}(\cdot), F_{22}(\mathbf{1}) \rangle.$$

(The equality $\varphi_{12} = \overline{\varphi_{21}}$ follows from the positivity of φ.) Further,

$$\int_\pi (\varphi_{11} - \varphi_{22}) \, d\nu = \varphi(\pi\varepsilon_{11} - \pi\varepsilon_{22}) = \varphi(\pi\varepsilon_{12}\varepsilon_{12}^*) - \varphi(\pi\varepsilon_{12}^*\varepsilon_{12})$$

$$= \varphi(\pi\varepsilon_{12}\varepsilon_{12}^*) + \psi(\pi\varepsilon_{12}^*\varepsilon_{12}) - \varrho(\pi\varepsilon_{12}^*\varepsilon_{12})$$
$$= \langle F(\pi\varepsilon_{12}^*), F(\pi\varepsilon_{12}^*) \rangle - \varrho(\pi\varepsilon_{22})$$
$$= \|F(\pi\varepsilon_{21})\|^2 - \varrho(\pi\varepsilon_{22}) = \langle F_{21}(\pi), F_{21}(\mathbf{1}) \rangle - \int_\pi \varrho_{22} \, d\nu$$

$$= \int_\pi (r_{21} - \varrho_{22}) \, d\nu.$$

(The function r_{21} is defined by (20).) Putting $\phi \equiv \varphi_{11}$ we have

$$\varphi = \begin{pmatrix} \phi & \varphi_{12} \\ \overline{\varphi_{12}} & \phi + \varrho_{22} - r_{21} \end{pmatrix}, \qquad \psi = \begin{pmatrix} \varrho_{11} - \phi & \varrho_{12} - \varphi_{12} \\ \overline{\varrho_{12}} - \overline{\varphi_{12}} & r_{21} - \phi \end{pmatrix}. \qquad (23)$$

It should be noted that all elements of matrices φ and ψ, with the exception of ϕ, are defined as the Radon-Nikodym derivatives associated with the fields F_{ij}. The function $\phi \in L^1(\Omega, \nu)^+$ necessarily satisfies inequalities

$$\max\{0, r_{21}(\omega) - \varrho_{22}(\omega)\} \leq \phi(\omega) \leq \min\{\varrho_{11}(\omega), r_{21}(\omega)\} \quad \text{a. e.}, \qquad (24)$$

$$|\varphi_{12}(\omega)| \leq [\phi(\omega)(\phi(\omega) + \varrho_{22}(\omega) - r_{21}(\omega))]^{1/2} \quad \text{a. e.}, \qquad (25)$$

$$|\varrho_{12}(\omega) - \varphi_{12}(\omega)| \leq [(\rho_{11}(\omega) - \phi(\omega))(r_{21}(\omega) - \phi(\omega))]^{1/2} \quad \text{a. e.} \qquad (26)$$

Theorem 4. *Let $\mathcal{N} = \mathcal{M} \otimes M_2$ be a W^*-algebra of type I_2. Then every w^*-continuous OVF $F : \mathcal{N} \to H$ is stationary.*

The following lemma gives conditionally ahead of time proof of the theorem.

Lemma 5. *If the function $\phi \in L^1(\Omega, \nu)$ satisfies inequalities (24) – (26), then the equality (22) holds when φ and ψ are defined by matrices (23).*

Proof. As F is linear and w^*-continuous (and by Prop. 3(iii)) we must verify only that the following equalities hold

$$\langle F(\pi\varepsilon_{ij}), F(\varepsilon_{kl})\rangle = \varphi(\varepsilon_{lk}(\pi\varepsilon_{ij})) + \psi((\pi\varepsilon_{ij})\varepsilon_{lk}), \ \pi \in \mathcal{M}^{\mathrm{pr}}, \ i,j,k,l \in \{1,2\}.$$

They are easily verified directly. For example, in view of (17),(8),(9) and by Prop. 1(iv) we have

$$\langle F(\pi\varepsilon_{ii}), F(\varepsilon_{ii})\rangle = \int_\pi \varrho_{ii}\, d\nu = \int_\pi (\varphi_{ii} + \psi_{ii})\, d\nu = \varphi(\pi\varepsilon_{ii}) + \psi(\pi\varepsilon_{ii})$$

$$= \varphi(\varepsilon_{ii}(\pi\varepsilon_{ii})) + \psi((\pi\varepsilon_{ii})\varepsilon_{ii}), \quad i = 1,2,$$

$$\langle F(\pi\varepsilon_{11}), F(\varepsilon_{12})\rangle = \langle F_{11}(\pi), F_{12}(\mathbf{1})\rangle = \langle F_{11}(\mathbf{1}), F_{12}(\pi)\rangle$$

$$= \langle F_{21}(\pi), F_{22}(\mathbf{1})\rangle = \int_\pi \varphi_{12}\, d\nu = \varphi(\pi\varepsilon_{21})$$

$$= \varphi((\pi\varepsilon_{21})\varepsilon_{11}) + \psi(\varepsilon_{11}(\pi\varepsilon_{21})).$$

Similarly we can verify remaining equalities, and the lemma follows. $\square$

Lemma 6. *Any OVF over the factor of type I_2 is stationary.*

Proof. Let $F : M_2 \to H$ be a OVF. Without loss of generality, we may assume that $\|F(\mathbf{I})\| = 1$. Let us denote for brevity $F_{ij} = F(\varepsilon_{ij})$. By virtue of standard arguments connected with unitary invariance, it is sufficient to consider the case when the matrix $\varrho = (\varrho_{ij})$ (here it is scalar) is diagonal:

$$\varrho = \begin{pmatrix} \varrho_{11} & 0 \\ 0 & \varrho_{22} \end{pmatrix}, \quad \varrho_{11} + \varrho_{22} = 1, \ \varrho_{ii} \geq 0.$$

Also note that

$$r_{ij} = \|F_{ij}\|^2, \quad r_{12} + r_{21} = 1.$$

Case 1: the rank of matrix ϱ equals 1. Let $\varrho_{11} = 1, \varrho_{22} = 0$ (the same arguments can be used if $\varrho_{11} = 0, \varrho_{22} = 1$). By the equality $\varrho_{12} = 0$ it follows that $\{F_{ij}\}$ is the orthogonal system of vectors in H. Now we see that equality (22) is satisfied with

$$\varphi = \begin{pmatrix} r_{21} & 0 \\ 0 & 0 \end{pmatrix}, \qquad \psi = \begin{pmatrix} r_{12} & 0 \\ 0 & 0 \end{pmatrix}$$

In fact, for $x = (x_{ij})$, $y = (y_{ij}) \in M_2$ we have

$$
\begin{aligned}
\langle F(x), F(y) \rangle &= \sum_{i,j=1}^{2} x_{ij}\overline{y_{ij}} \|F_{ij}\|^2 = x_{11}\overline{y_{11}} + r_{12}x_{12}\overline{y_{12}} + r_{21}x_{21}\overline{y_{21}} \\
&= r_{21}(x_{11}\overline{y_{11}} + x_{21}\overline{y_{21}}) + r_{12}(x_{11}\overline{y_{11}} + x_{12}\overline{y_{12}}) \\
&= \mathrm{Tr}\left[\begin{pmatrix} r_{21} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \overline{y_{11}} & \overline{y_{21}} \\ \overline{y_{12}} & \overline{y_{22}} \end{pmatrix} \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \right] \\
&+ \mathrm{Tr}\left[\begin{pmatrix} r_{12} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \begin{pmatrix} \overline{y_{11}} & \overline{y_{21}} \\ \overline{y_{12}} & \overline{y_{22}} \end{pmatrix} \right] \\
&= \varphi(y^*x) + \psi(xy^*).
\end{aligned}
$$

It is easily seen that φ and ψ are determined uniquely.

Case 2: the rank of matrix ϱ equals 2. In the case

$$
\varrho = \begin{pmatrix} \varrho_{11} & 0 \\ 0 & \varrho_{22} \end{pmatrix}, \quad \varrho_{11} + \varrho_{22} = 1, \ \varrho_{ii} > 0,
$$

and the required matrices φ and ψ obtain the form

$$
\varphi = \begin{pmatrix} \phi & \varphi_{12} \\ \overline{\varphi_{12}} & \phi + \varrho_{22} - r_{21} \end{pmatrix}, \quad \psi = \begin{pmatrix} \varrho_{11} - \phi & -\varphi_{12} \\ -\overline{\varphi_{12}} & r_{21} - \phi \end{pmatrix},
$$

where

$$
\varphi_{12} = \langle F_{21}, F_{22} \rangle = -\langle F_{21}, F_{11} \rangle = \langle F_{11}, F_{12} \rangle \tag{27}
$$

(the second equality follows from $\varrho_{12} = 0$ (see (22)), the third one does from Prop. 1(iv), and ϕ is an unknown for now parameter satisfying inequalities (see (24) – (26))

$$
\max\{0, r_{21} - \varrho_{22}\} \le \phi \le \min\{\varrho_{11}, r_{21}\}, \tag{28}
$$

$$
|\varphi_{12}|^2 \le \phi(\phi + \varrho_{22} - r_{21}), \tag{29}
$$

$$
|\varphi_{12}|^2 \le (\rho_{11} - \phi)(r_{21} - \phi). \tag{30}
$$

We show that $\phi_0 = \frac{1}{2}[r_{21} - \varrho_{22} + \sqrt{(r_{21} - \varrho_{22})^2 + 4|\varphi_{12}|^2}]$, a solution of the quadratic equation $\phi(\phi + \varrho_{22} - r_{21}) - |\varphi_{12}|^2 = 0$, satisfies inequalities (28) – (30). It is evident that ϕ_0 satisfies (29) and $\phi_0 \ge 0$. By (27)

$$
2\phi_0 \le r_{21} - \varrho_{22} + \sqrt{(r_{21} - \varrho_{22})^2 + 4r_{21}\varrho_{22}} = 2r_{21},
$$

$$
\begin{aligned}
2\phi_0 &\le \varrho_{11} - r_{12} + \sqrt{(r_{12} - \varrho_{11})^2 + 4|\varphi_{12}|^2} \le \varrho_{11} - r_{12} + \sqrt{(r_{12} - \varrho_{11})^2 + 4r_{12}\varrho_{11}} \\
&= 2\varrho_{11}.
\end{aligned}
$$

Hence we have $\phi_0 \le \min\{\varrho_{11}, r_{21}\}$ and (28) follows.

In order to verify (30) we represent F in the coordinate form. As $\dim \operatorname{rg} F \leq 4$ we will assume that F acts into $\mathbb{C}^4$. By a suitable choice of the coordinate system in $\mathbb{C}^4$ we may assume that

$$F(I) = (1, 0, 0, 0), \quad F_{11} = (\alpha, \zeta, 0, 0), \quad F_{22} = (1 - \alpha, -\zeta, 0, 0).$$

In this case

$$0 < \alpha < 1, \quad |\zeta|^2 = \alpha - \alpha^2, \ \zeta = \omega|\zeta|, \ |\omega| = 1.$$

Put $F_{12} = (\xi_1, \xi_2, \xi_3, \xi_4)$, $F_{21} = (\eta_1, \eta_2, \eta_3, \eta_4)$ and note that

$$\eta_1 = \langle F(\varepsilon_{21}), F(I) \rangle = \operatorname{Tr}(\varrho\varepsilon_{21}) = 0,$$

Similarly, $\xi_1 = 0$. In addition, $\eta_2 \overline{\zeta} = -\overline{\xi_2}\zeta$ (by Prop. 1(iv)) so that $\eta_2 = -\omega^2 \overline{\xi_2}$. Thus, putting $\xi \equiv \xi_2$,

$$F_{12} = (0, \xi, \xi_3, \xi_4), \quad F_{21} = (0, -\omega^2\overline{\xi}, \eta_3, \eta_4),$$

and also according to Prop. 1(i),(ii),

$$2|\xi|^2 + |\xi_3|^2 + |\xi_4|^2 + |\eta_3|^2 + |\eta_4|^2 = 1, \quad \xi_3\overline{\eta_3} + \xi_4\overline{\eta_4} = \overline{\omega}^2\xi^2.$$

It should be noted that

$$\begin{aligned}
r_{12}r_{21} &= (|\xi|^2 + |\xi_3|^2 + |\xi_4|^2)(|\xi|^2 + |\eta_3|^2 + |\eta_4|^2) \\
&= |\xi|^2(1 - |\xi|^2) + (|\xi_3|^2 + |\xi_4|^2)(|\eta_3|^2 + |\eta_4|^2) \\
&\geq |\xi|^2 - |\xi|^4 + |\xi_3\overline{\eta_3} + \xi_4\overline{\eta_4}|^2 = |\xi|^2.
\end{aligned}$$

Therefore,

$$|\varphi_{12}|^2 = |\langle F_{21}, F_{11} \rangle|^2 = |\xi|^2|\zeta|^2 \leq r_{12}r_{21}\varrho_{11}\varrho_{22}.$$

Since $r_{12} + r_{21} = \varrho_{11} + \varrho_{22} = 1$, we have

$$\varrho_{22} - r_{21} = \varrho_{22}r_{12} - \varrho_{11}r_{21}.$$

Now (30) follows from computation:

$$\begin{aligned}
2[(\varrho_{11} - \phi_0)(r_{21} - \phi_0) - |\varphi_{12}|^2] &= 2(\varrho_{11}r_{22} - \phi_0) \\
&= 2\varrho_{11}r_{21} - r_{21} + \varrho_{22} - \sqrt{(\varrho_{22} - r_{21})^2 + 4|\varphi_{12}|^2} \\
&= \varrho_{11}r_{21} + \varrho_{22}r_{12} - \sqrt{(\varrho_{11}r_{21} - \varrho_{22}r_{12})^2 + 4|\varphi_{12}|^2} \\
&\geq \varrho_{11}r_{21} + \varrho_{22}r_{12} - \sqrt{(\varrho_{11}r_{21} - \varrho_{22}r_{12})^2 + 4r_{12}r_{21}\varrho_{11}\varrho_{22}} = 0.
\end{aligned}$$

Applying Lemma 5 to the case in question, we finish the proof. $\qquad\square$

Proof of Theorem 4. in view of lemma 5 it is sufficient to show the existence of a function $\phi \in L^1(\Omega, \nu)$ with the properties set out in (24) – (26). It is convenient to begin with the atomic case. Let $\Delta = \{\pi_k\}_{k \in K}$ be an arbitrary measurable partition of Ω, i. e.

$$\Omega = \bigcup_k \pi_k, \quad \pi_k \cap \pi_l = \emptyset \ (k \neq l).$$

Via $\mathcal{M}_\Delta$ we denote the algebra of functions $x \in L^\infty(\Omega, \nu)$ in the form of $x = \sum_k \lambda_k \pi_k$, $\lambda_k \in \mathbb{C}$, $\pi_k \in \Delta$. Let $\mathcal{N}_\Delta = \mathcal{M}_\Delta \otimes M_2$ be a subalgebra of $\mathcal{N}$, being the direct sum of factors of type I_2:

$$\mathcal{N}_\Delta = \sum_k \pi_k \otimes M_2 \quad \text{where } \pi \otimes M_2 \equiv \left\{ \begin{pmatrix} \lambda_{11}\pi & \lambda_{12}\pi \\ \lambda_{21}\pi & \lambda_{22}\pi \end{pmatrix} : \lambda_{ij} \in \mathbb{C} \right\},$$

Particularly $\mathcal{N}_\Delta$ is a W^*-algebra. The restriction $F\,|\,\mathcal{N}_\Delta$ is a w^*-continuous OVF over $\mathcal{N}_\Delta$. In this case the parametric functions characterizing $F\,|\,\mathcal{N}_\Delta$ are defined as follows

$$\varrho_{ii}^\Delta = \sum_k \varrho_{iik}^\Delta \pi_k, \quad \varrho_{iik}^\Delta = \langle F_{ii}(\pi_k), F_{ii}(\mathbf{1})\rangle,$$

$$\varrho_{ij}^\Delta = \sum_k \varrho_{ijk}^\Delta \pi_k, \quad \varrho_{ijk}^\Delta = \langle F_{ji}(\pi_k), F_{11}(\mathbf{1}) + F_{22}(\mathbf{1})\rangle, \ i \neq j,$$

$$r_{ij}^\Delta = \sum_k r_{ijk}^\Delta \pi_k, \quad r_{ijk}^\Delta = \|F_{ji}(\pi_k)\|^2,$$

$$\varphi_{12}^\Delta = \overline{\varphi_{21}^\Delta} = \sum_k \varphi_{12k}^\Delta \pi_k, \quad \varphi_{12k}^\Delta = \langle F_{21}(\pi_k), F_{11}(\mathbf{1})\rangle,$$

with

$$r_{12}^\Delta + r_{21}^\Delta = \varrho_{11}^\Delta + \varrho_{22}^\Delta.$$

According to Lemma 6 and (18) – (20) it is defined a function $\phi^\Delta \in L^1(\Omega, \nu)$ such that

$$\max\{0, r_{21}^\Delta(\omega) - \varrho_{22}^\Delta(\omega)\} \leq \phi^\Delta(\omega) \leq \min\{\varrho_{11}^\Delta(\omega), r_{21}^\Delta(\omega)\} \text{ a. e.,} \qquad (31)$$

$$|\varphi_{12}^\Delta(\omega)| \leq [\phi^\Delta(\omega)(\phi^\Delta(\omega) + \varrho_{22}^\Delta(\omega) - r_{21}^\Delta(\omega))]^{1/2} \text{ a. e.,}$$

$$|\varrho_{12}^\Delta(\omega) - \varphi_{12}^\Delta(\omega)| \leq [\rho_{11}^\Delta(\omega) - \phi^\Delta(\omega)]^{1/2}[r_{21}^\Delta(\omega) - \phi^\Delta(\omega)]^{1/2} \text{ a. e.}$$

Let us turn to the general case. Since (Ω, ν) is a localizable measure space, for the construction of a function $\phi \in L^1(\Omega, \nu)^+$ we can restrict our attention to the case when $\nu(\Omega) < +\infty$. We will choose a sequence $\Delta_n = \{\pi_{kn}\}_k$ of partitions Ω such that

$$|\varrho_{11}(\omega) - \varrho_{11k}^n|, \ |r_{21}(\omega) - r_{21k}^n|, \ ||\varphi_{12}(\omega)| - |\varphi_{12k}^n|| < \frac{1}{n}, \quad \omega \in \pi_{kn}.$$

(Here, for example, $\varrho_{11k}^n \equiv \varrho_{11k}^{\Delta_n}$.) The partition $\{\pi_{kn}\}_k$ may be constructed in the following way: denoting

$$\sigma_{tn} = \{\omega : \frac{t}{n} \leq \varrho_{11}(\omega) < \frac{t+1}{n}\},$$

$$\sigma_{tn}' = \{\omega : \frac{t}{n} \leq r_{21}(\omega) < \frac{t+1}{n}\},$$

$$\sigma_{tn}'' = \{\omega : \frac{t}{n} \leq |\varphi_{21}(\omega)| < \frac{t+1}{n}\}, \quad t = 0, 1, 2, ...,$$

we put for multi-index $k = (t, s, m)$, $t, s, m = 0, 1, 2, ...$

$$\pi_{kn} = \sigma_{tn} \cap \sigma'_{sn} \cap \sigma''_{mn}, \quad \text{if } \nu(\pi_{kn}) > 0.$$

As $k = (t, s, m)$ we have

$$\frac{t}{n} \leq \varrho_{11}(\omega) < \frac{t+1}{n}, \quad \omega \in \pi_{kn} \Rightarrow$$

$$\frac{t}{n}\nu(\pi_{kn}) \leq \int\limits_{\pi_{kn}} \varrho_{11} \, d\nu = \varrho_{11k}^n \nu(\pi_{kn}) \leq \frac{t+1}{n}\nu(\pi_{kn}) \Rightarrow$$

$$\frac{t}{n} \leq \varrho_{11k}^n \leq \frac{t+1}{n} \quad \Rightarrow \quad |\varrho_{11}(\omega) - \varrho_{11k}^n| < \frac{1}{n} \quad (\omega \in \pi_{kn}).$$

The arbitrariness of k and similar computations for r_{21}, $|\varphi_{12}|$ then implies

$$|\varrho_{11}(\omega) - \varrho_{11k}^n|, \; |r_{21}(\omega) - r_{21k}^n|, \; ||\varphi_{12}(\omega)| - |\varphi_{12k}^n|| < \frac{1}{n}, \quad \text{a. e.}$$

Thus, $\varrho_{11}^n, r_{21}^n, |\varphi_{12}^n|$ are the sequences of simple integrable functions that converge uniformly (respectively) to $\varrho_{11}, r_{21}, |\varphi_{12}|$. As a consequence, $\varrho_{22}^n \rightrightarrows \varrho_{22}, r_{12}^n \rightrightarrows r_{12}$. In accordance with Lemma 6 we put

$$\phi^{\Delta_n}(\omega) = \frac{1}{2}[r_{21}^n(\omega) - \varrho_{22}^n(\omega) + \sqrt{(r_{21}^n(\omega) - \varrho_{22}^n(\omega))^2 + 4|\varphi_{12}^n(\omega)|^2}].$$

By (31) the sequence ϕ^{Δ_n} is dominated with the integrable function $\min\{\varrho_{11}, r_{21}\}$ (we keep in mind that $\nu(\Omega) < +\infty$). So by the Dominated Convergence Theorem, the function $\phi(\omega) \equiv \lim_n \phi^{\Delta_n}(\omega)$, $\omega \in \Omega$, is integrable and satisfies inequalities (24) – (26). The proof is complete. $\square$

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* Department of Mathematical Analysis
 Kazan Federal University
 Kremlyovskaya St., 18
 Kazan 420008
 RUSSIAN FEDERATION
 E-mail address: glugovay@kpfu.ru
 E-mail address: Anatolij.Sherstnev@ksu.ru